Topology in Photonic Graphene

Pierre Wulles - 12/06/2023

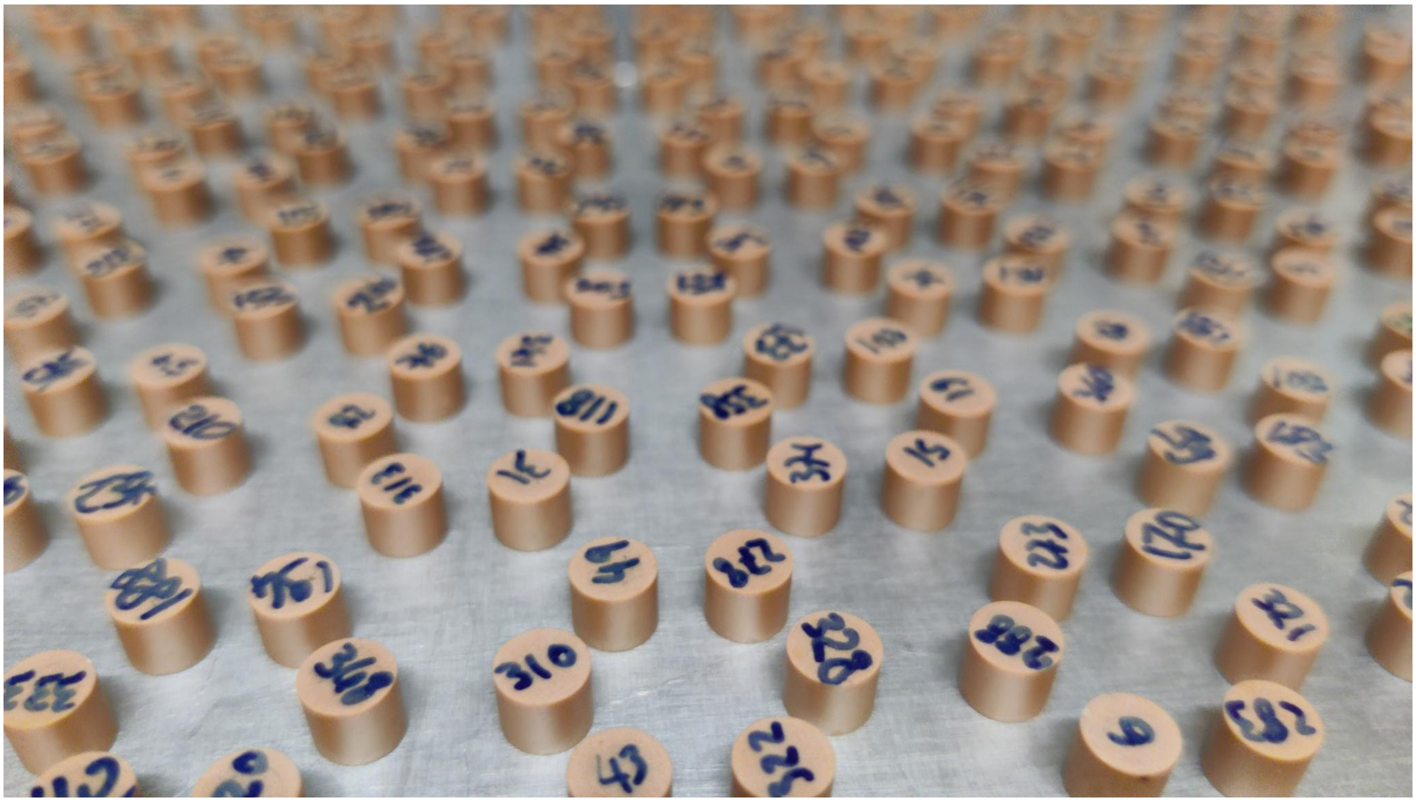


Figure 1.

Vidéo

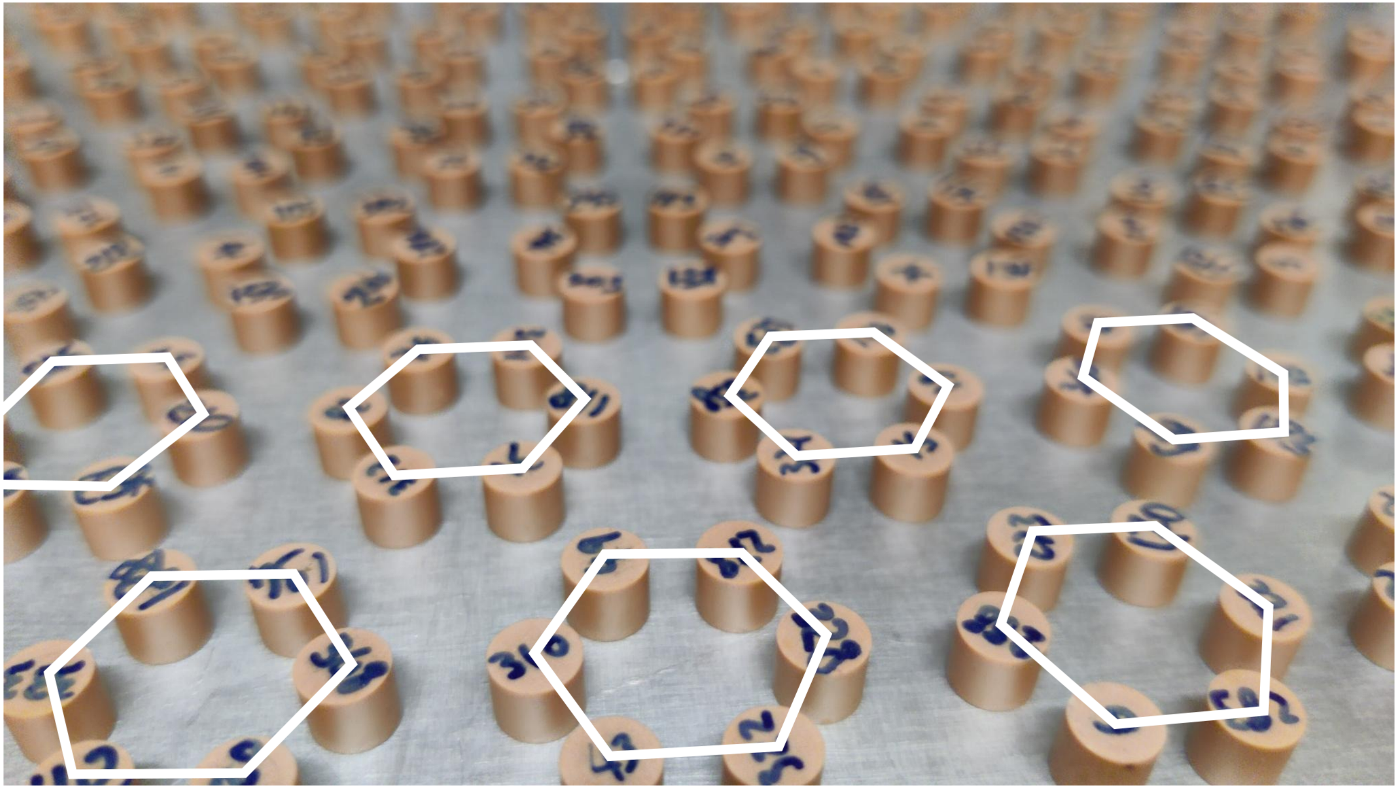


Figure 2.

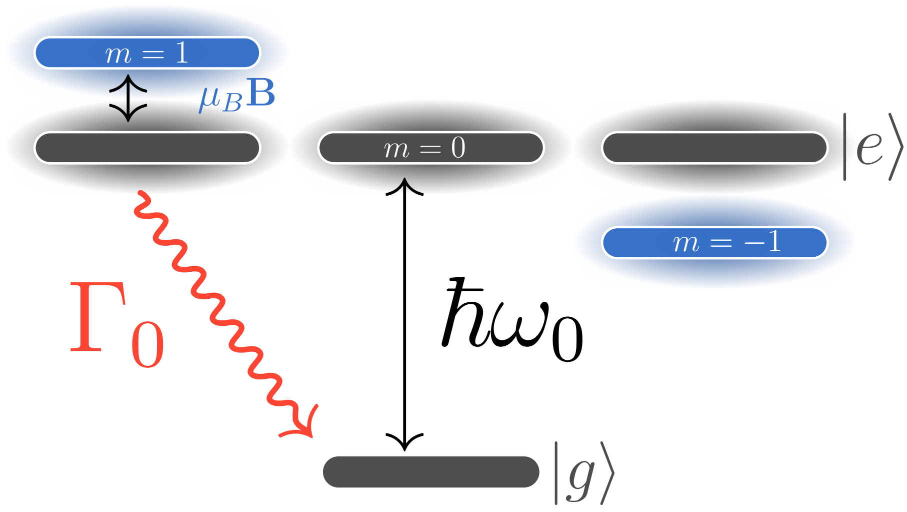


Figure 3.

For one atom, the two-level system hamiltonian can be written as:

$$H_{\text{atom}} = \sum_{m=-1}^{1} \left(\hbar\omega_0 + m\mu_B B - i\frac{\hbar\Gamma_0}{2} \right) |e_m\rangle\langle e_m|$$

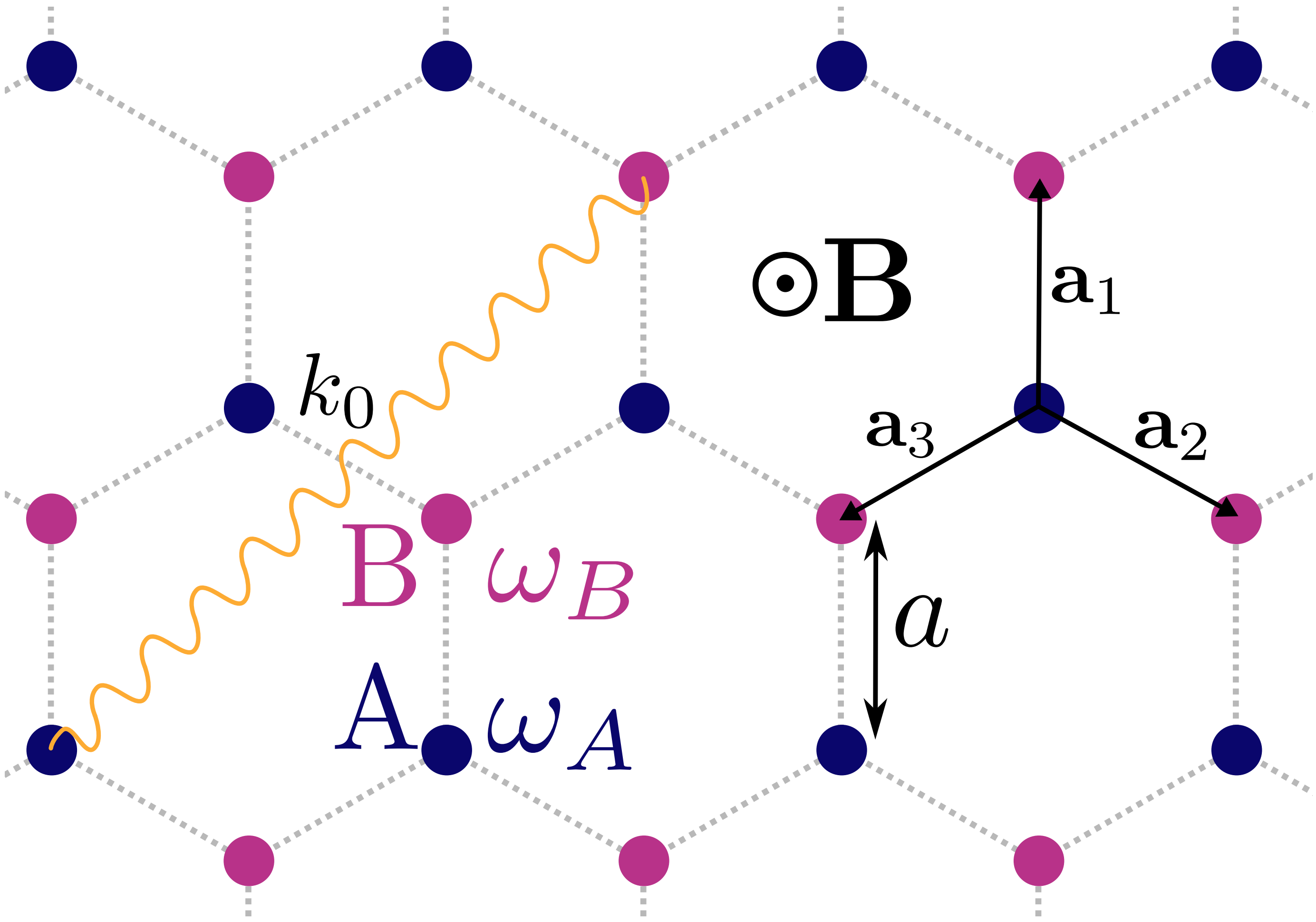

k_0
$\odot \mathbf{B}$
$\mathbf{a}_1$
$\mathbf{a}_2$
$\mathbf{a}_3$
a
B ω_B
A ω_A

Figure 4.

Our Hamiltonian

$$
\begin{aligned}
H \; = \; & \sum_{p=1}^{N} \sum_{m=-1}^{1} \left(\hbar\omega_0^{A,B} + m\mu_B B - i\frac{\hbar\Gamma_0}{2} \right) |e_{p,m}\rangle\langle e_{p,m}| \\
+ \; & \frac{3\pi\hbar\Gamma_0}{k_0} \sum_{p\neq q}^{N} \sum_{m_p,m_q=-1}^{1} \mathcal{G}_{m_p m_q}(\mathbf{r}_p - \mathbf{r}_q) |e_{p,m_p}\rangle\langle e_{q,m_q}|
\end{aligned}
$$

H can be seen as a function of three parameters:

- $\Delta_{AB} = (\omega_0^B - \omega_0^A)/2\Gamma_0$ (Frequency detuning) $\rightarrow$ **inversion symmetry breaking**
- $\Delta_{\mathrm{B}} = \mu B/\Gamma_0$ (Zeeman shift) $\rightarrow$ **time reversal symmetry breaking**
- $k_0 a = (\omega_0/c)a$ (Spacing)

But I must insist on the following:

- No nearest-neighbour model
- No crystal (no "bonds" between sites)
- Sites B and magnetic field B are two unrelated things
- H is non-hermitian

Our Hamiltonian

$$
\begin{aligned}
H \; = \; & \sum_{p=1}^{N} \sum_{m=-1}^{1} \left(\hbar\omega_0^{A,B} + m\mu_B B - i\frac{\hbar\Gamma_0}{2} \right) |e_{p,m}\rangle\langle e_{p,m}| \\
+ \; & \frac{3\pi\hbar\Gamma_0}{k_0} \sum_{p\neq q}^{N} \sum_{m_p,m_q=-1}^{1} \mathcal{G}_{m_p m_q}(\mathbf{r}_p - \mathbf{r}_q) |e_{p,m_p}\rangle\langle e_{q,m_q}|
\end{aligned}
$$

H can be seen as a function of three parameters:

- $\Delta_{AB} = (\omega_0^B - \omega_0^A)/2\Gamma_0$ (Frequency detuning) $\rightarrow$ **inversion symmetry breaking**
- $\Delta_{\mathrm{B}} = \mu B/\Gamma_0$ (Zeeman shift) $\rightarrow$ **time reversal symmetry breaking**
- $k_0 a = (\omega_0/c)a$ (Spacing)

But I must insist on the following:

- No nearest-neighbour model
- No crystal (no “bonds” between sites)
- Sites B and magnetic field $\mathbf{B}$ are two unrelated things
- H is non-hermitian

Our Hamiltonian

$$H = \sum_{p=1}^{N} \sum_{m=-1}^{1} \left(\hbar\omega_0^{A,B} + m\mu_B B - i\frac{\hbar\Gamma_0}{2} \right) |e_{p,m}\rangle\langle e_{p,m}| + \frac{3\pi\hbar\Gamma_0}{k_0} \sum_{p\neq q}^{N} \sum_{m_p,m_q=-1}^{1} \mathcal{G}_{m_p m_q}(\mathbf{r}_p - \mathbf{r}_q) |e_{p,m_p}\rangle\langle e_{q,m_q}|$$

H can be seen as a function of three parameters:

- $\Delta_{AB} = (\omega_0^B - \omega_0^A)/2\Gamma_0$ (Frequency detuning) $\rightarrow$ **inversion symmetry breaking**
- $\Delta_\mathbf{B} = \mu B/\Gamma_0$ (Zeeman shift) $\rightarrow$ **time reversal symmetry breaking**
- $k_0 a = (\omega_0/c)a$ (Spacing)

But I must insist on the following:

- No nearest-neighbour model
- No crystal (no "bonds" between sites)
- Sites B and magnetic field $\mathbf{B}$ are two unrelated things
- H is non-hermitian

Our Hamiltonian

$$H = \sum_{p=1}^{N} \sum_{m=-1}^{1} \left(\hbar\omega_0^{A,B} + m\mu_B B - i\frac{\hbar\Gamma_0}{2} \right) |e_{p,m}\rangle\langle e_{p,m}| + \frac{3\pi\hbar\Gamma_0}{k_0} \sum_{p\neq q}^{N} \sum_{m_p,m_q=-1}^{1} \mathcal{G}_{m_p m_q}(\mathbf{r}_p - \mathbf{r}_q) |e_{p,m_p}\rangle\langle e_{q,m_q}|$$

H can be seen as a function of three parameters:

- $\Delta_{AB} = (\omega_0^B - \omega_0^A)/2\Gamma_0$ (Frequency detuning) $\rightarrow$ **inversion symmetry breaking**
- $\Delta_{\mathbf{B}} = \mu B/\Gamma_0$ (Zeeman shift) $\rightarrow$ **time reversal symmetry breaking**
- $k_0 a = (\omega_0/c)a$ (Spacing)

But I must insist on the following:

- No nearest-neighbour model
- No crystal (no "bonds" between sites)
- Sites B and magnetic field $\mathbf{B}$ are two unrelated things
- H is non-hermitian

Our Hamiltonian

$$H = \sum_{p=1}^{N} \sum_{m=-1}^{1} \left(\hbar\omega_0^{A,B} + m\mu_B B - i\frac{\hbar\Gamma_0}{2} \right) |e_{p,m}\rangle\langle e_{p,m}| + \frac{3\pi\hbar\Gamma_0}{k_0} \sum_{p\neq q}^{N} \sum_{m_p,m_q=-1}^{1} \mathcal{G}_{m_p m_q}(\mathbf{r}_p - \mathbf{r}_q) |e_{p,m_p}\rangle\langle e_{q,m_q}|$$

H can be seen as a function of three parameters:

- $\Delta_{AB} = (\omega_0^B - \omega_0^A)/2\Gamma_0$ (Frequency detuning) $\rightarrow$ **inversion symmetry breaking**
- $\Delta_{\mathbf{B}} = \mu B/\Gamma_0$ (Zeeman shift) $\rightarrow$ **time reversal symmetry breaking**
- $k_0 a = (\omega_0/c)a$ (Spacing)

But I must insist on the following:

- No nearest-neighbour model
- No crystal (no "bonds" between sites)
- Sites B and magnetic field $\mathbf{B}$ are two unrelated things
- H is non-hermitian

Our Hamiltonian

$$H = \sum_{p=1}^{N} \sum_{m=-1}^{1} \left(\hbar\omega_0^{A,B} + m\mu_B B - i\frac{\hbar\Gamma_0}{2} \right) |e_{p,m}\rangle\langle e_{p,m}| + \frac{3\pi\hbar\Gamma_0}{k_0} \sum_{p\neq q}^{N} \sum_{m_p,m_q=-1}^{1} \mathcal{G}_{m_p m_q}(\mathbf{r}_p - \mathbf{r}_q) |e_{p,m_p}\rangle\langle e_{q,m_q}|$$

H can be seen as a function of three parameters:

- $\Delta_{AB} = (\omega_0^B - \omega_0^A)/2\Gamma_0$ (Frequency detuning) $\rightarrow$ **inversion symmetry breaking**
- $\Delta_{\mathbf{B}} = \mu B/\Gamma_0$ (Zeeman shift) $\rightarrow$ **time reversal symmetry breaking**
- $k_0 a = (\omega_0/c)a$ (Spacing)

But I must insist on the following:

- No nearest-neighbour model
- No crystal (no "bonds" between sites)
- Sites B and magnetic field $\mathbf{B}$ are two unrelated things
- H is non-hermitian

Our Hamiltonian

$$H = \sum_{p=1}^{N}\sum_{m=-1}^{1}\left(\hbar\omega_0^{A,B} + m\mu_B B - i\frac{\hbar\Gamma_0}{2}\right)|e_{p,m}\rangle\langle e_{p,m}|$$
$$+ \frac{3\pi\hbar\Gamma_0}{k_0}\sum_{p\neq q}^{N}\sum_{m_p,m_q=-1}^{1}\mathcal{G}_{m_p m_q}(\mathbf{r}_p - \mathbf{r}_q)|e_{p,m_p}\rangle\langle e_{q,m_q}|$$

H can be seen as a function of three parameters:

- $\Delta_{AB} = (\omega_0^B - \omega_0^A)/2\Gamma_0$ (Frequency detuning) $\rightarrow$ **inversion symmetry breaking**
- $\Delta_{\mathbf{B}} = \mu B/\Gamma_0$ (Zeeman shift) $\rightarrow$ **time reversal symmetry breaking**
- $k_0 a = (\omega_0/c)a$ (Spacing)

But I must insist on the following:

- No nearest-neighbour model
- No crystal (no "bonds" between sites)
- Sites B and magnetic field $\mathbf{B}$ are two unrelated things
- H is non-hermitian

Our Hamiltonian

$$H = \sum_{p=1}^{N} \sum_{m=-1}^{1} \left(\hbar\omega_0^{A,B} + m\mu_B B - i\frac{\hbar\Gamma_0}{2} \right) |e_{p,m}\rangle\langle e_{p,m}| + \frac{3\pi\hbar\Gamma_0}{k_0} \sum_{p\neq q}^{N} \sum_{m_p,m_q=-1}^{1} \mathcal{G}_{m_p m_q}(\mathbf{r}_p - \mathbf{r}_q) |e_{p,m_p}\rangle\langle e_{q,m_q}|$$

H can be seen as a function of three parameters:

- $\Delta_{AB} = (\omega_0^B - \omega_0^A)/2\Gamma_0$ (Frequency detuning) $\longrightarrow$ **inversion symmetry breaking**
- $\Delta_{\mathbf{B}} = \mu B/\Gamma_0$ (Zeeman shift) $\longrightarrow$ **time reversal symmetry breaking**
- $k_0 a = (\omega_0/c)a$ (Spacing)

But I must insist on the following:

- No nearest-neighbour model
- No crystal (no “bonds” between sites)
- Sites B and magnetic field $\mathbf{B}$ are two unrelated things
- H is non-hermitian

Our Hamiltonian

$$H = \sum_{p=1}^{N} \sum_{m=-1}^{1} \left(\hbar\omega_0^{A,B} + m\mu_B B - i\frac{\hbar\Gamma_0}{2} \right) |e_{p,m}\rangle\langle e_{p,m}| + \frac{3\pi\hbar\Gamma_0}{k_0} \sum_{p\neq q}^{N} \sum_{m_p,m_q=-1}^{1} \mathcal{G}_{m_p m_q}(\mathbf{r}_p - \mathbf{r}_q) |e_{p,m_p}\rangle\langle e_{q,m_q}|$$

H can be seen as a function of three parameters:

- $\Delta_{AB} = (\omega_0^B - \omega_0^A)/2\Gamma_0$ (Frequency detuning) $\rightarrow$ **inversion symmetry breaking**
- $\Delta_{\mathbf{B}} = \mu B/\Gamma_0$ (Zeeman shift) $\rightarrow$ **time reversal symmetry breaking**
- $k_0 a = (\omega_0/c)a$ (Spacing)

But I must insist on the following:

- No nearest-neighbour model
- No crystal (no "bonds" between sites)
- Sites B and magnetic field $\mathbf{B}$ are two unrelated things
- H is non-hermitian

Our Brillouin zone

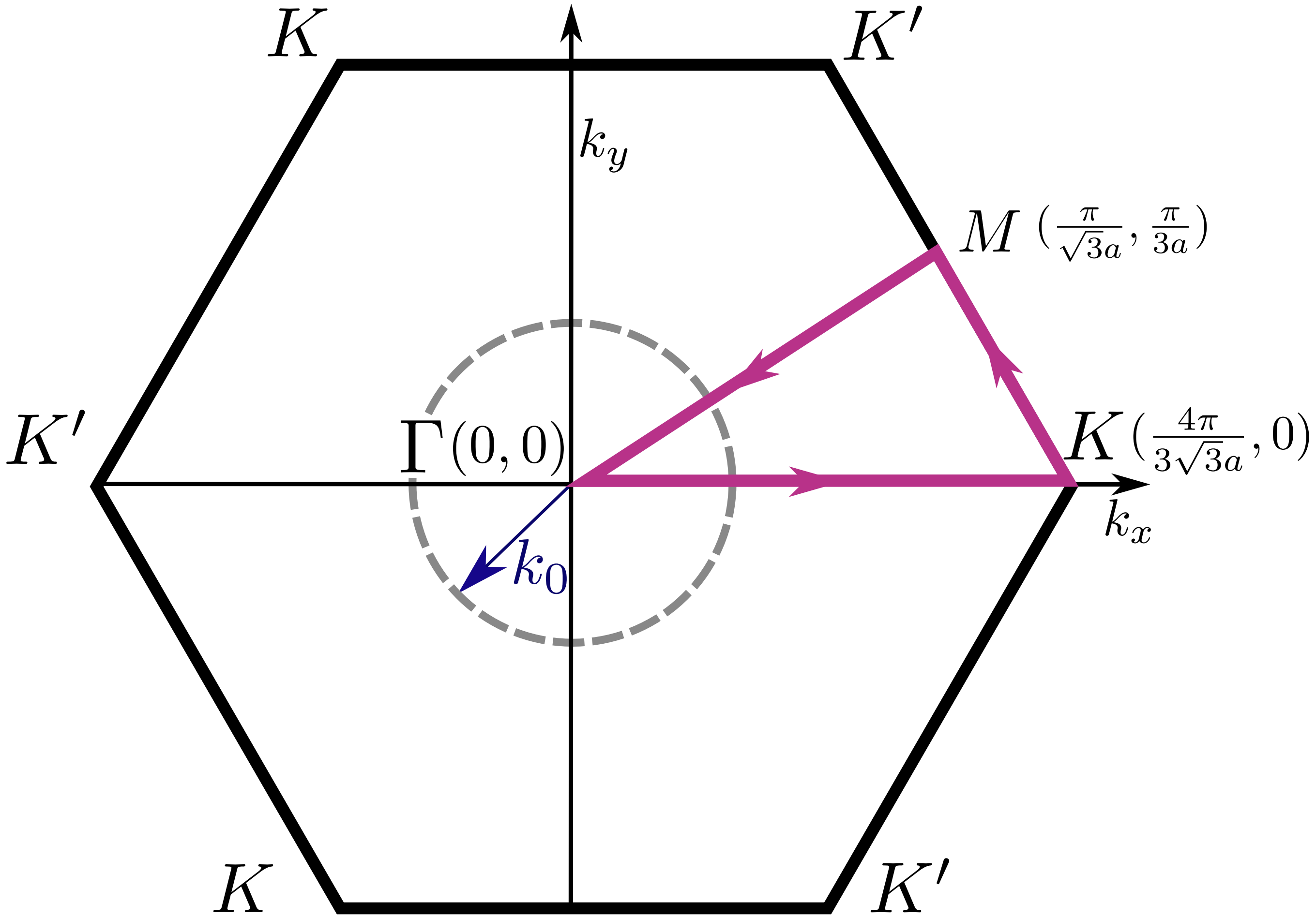
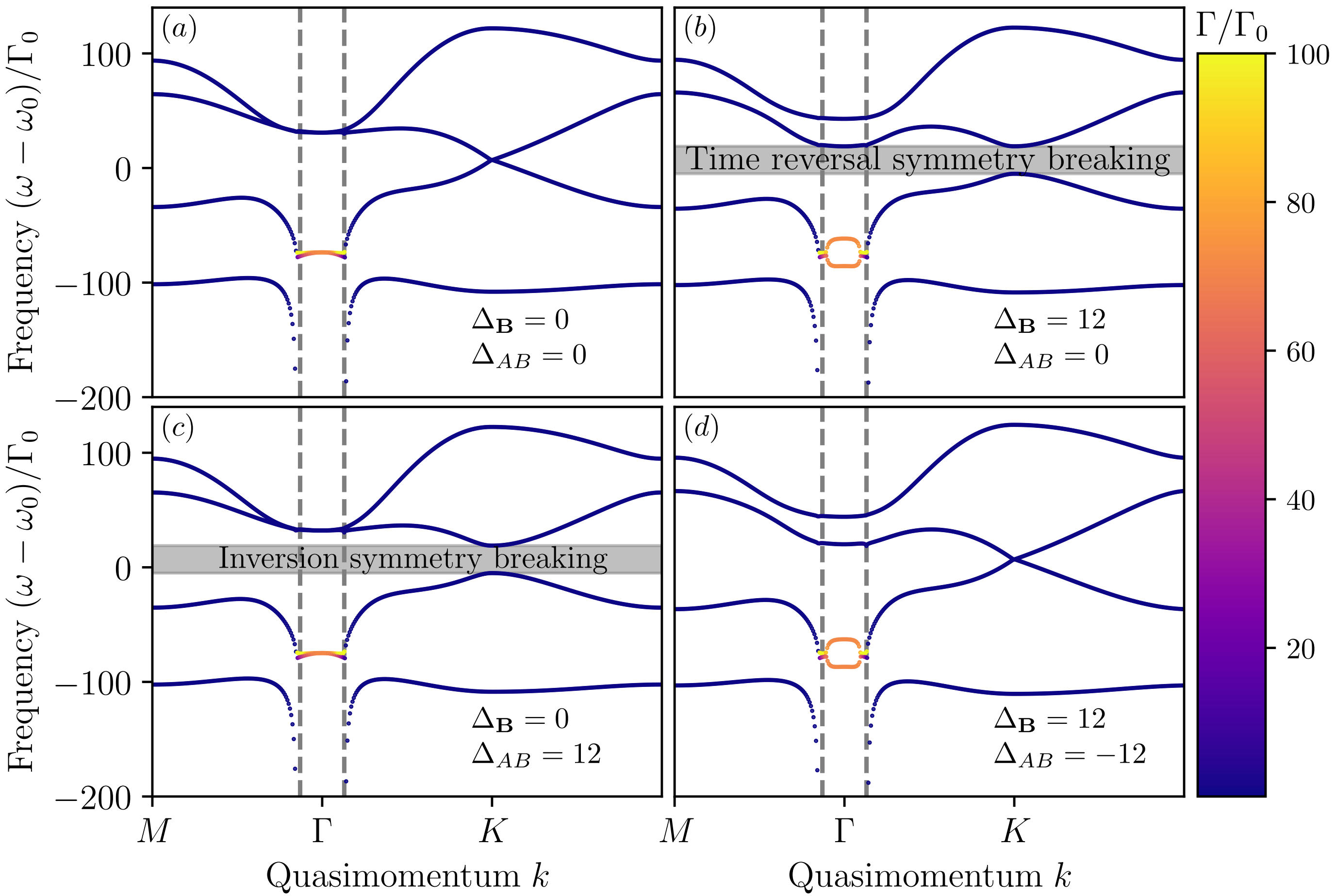


Figure 5.



(a)
(b)
(c)
(d)
Frequency $(\omega - \omega_0)/\Gamma_0$
100
0
−100
−200
Time reversal symmetry breaking
Inversion symmetry breaking
$\Delta_{\mathbf{B}} = 0$
$\Delta_{AB} = 0$
$\Delta_{\mathbf{B}} = 12$
$\Delta_{AB} = 0$
$\Delta_{\mathbf{B}} = 0$
$\Delta_{AB} = 12$
$\Delta_{\mathbf{B}} = 12$
$\Delta_{AB} = -12$
M
Γ
K
Quasimomentum k
Γ/Γ_0
100
80
60
40
20

Figure 6.

Width of the gap

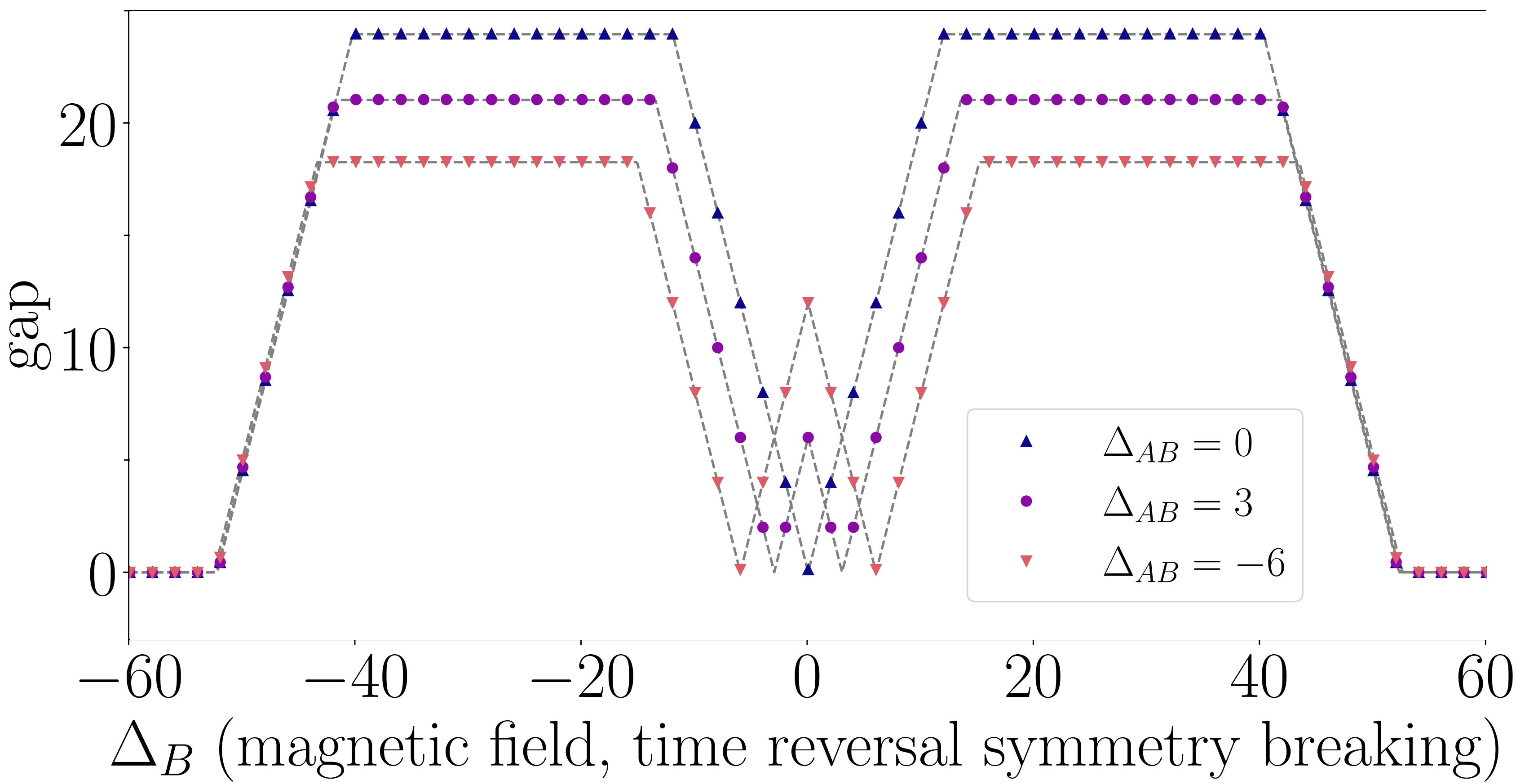


Figure 7.

Animation

Figure 8. Classification of objects *globally*

Localisation on sites A

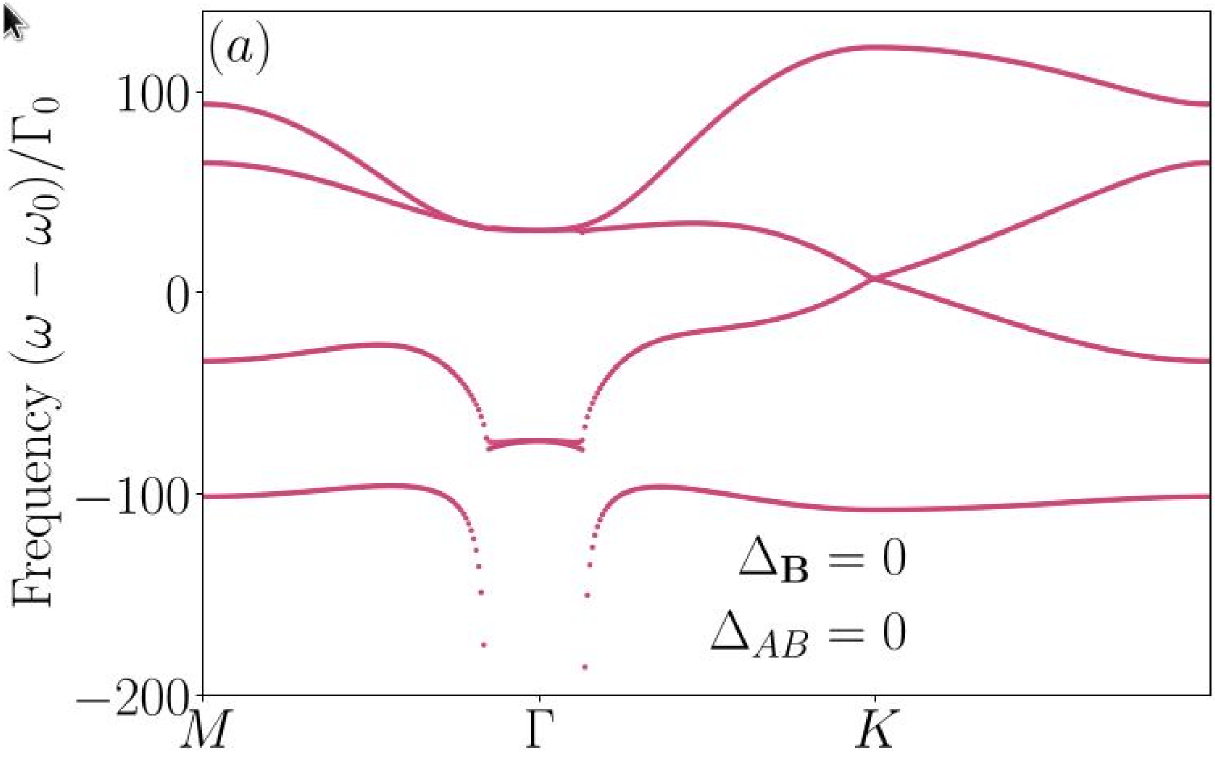


Figure 9.

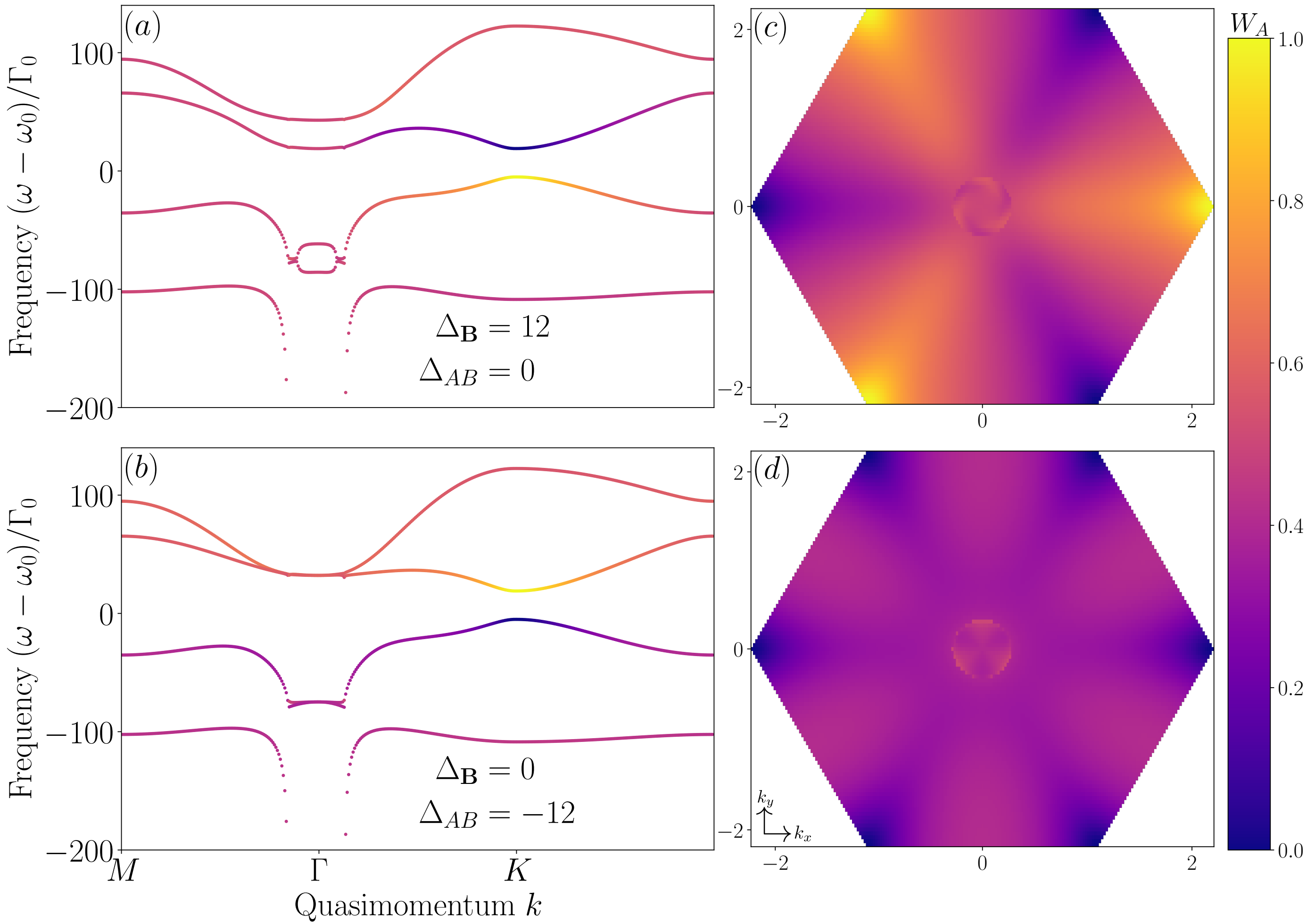
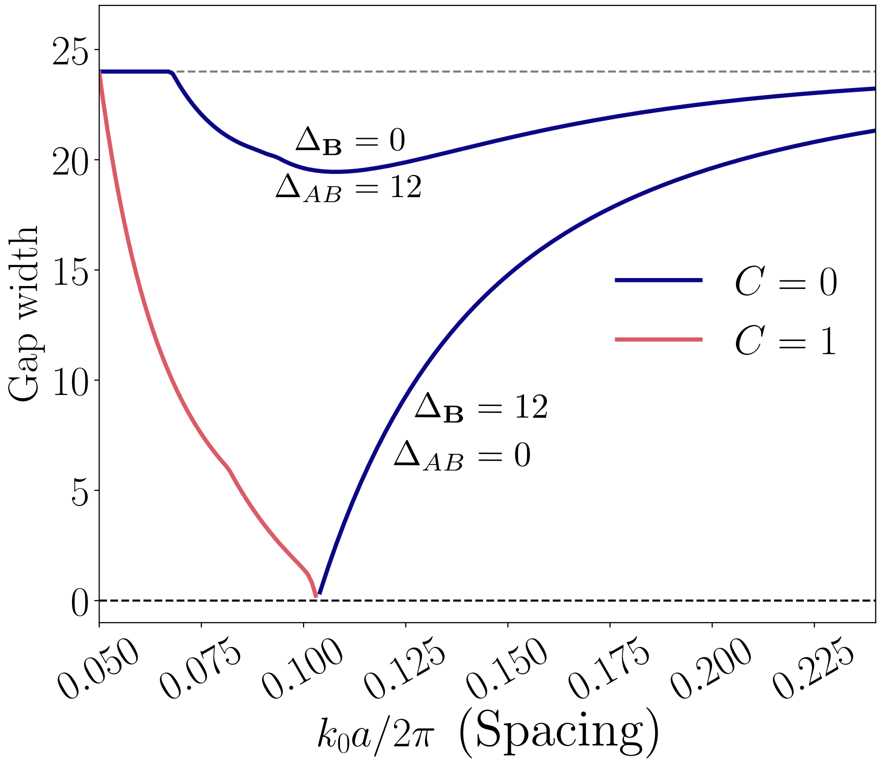


Figure 10. Localization on sites A

Closing the gap: another hint



$$C = \frac{1}{2\pi}\int_{\mathrm{BZ}} \Omega(\mathbf{k}) d^2\mathbf{k}$$

$$\Omega(\mathbf{k}) \;=\; i\Big[\Big\langle \frac{\partial\psi(\mathbf{k})}{\partial k_x}\Big|\frac{\partial\psi(\mathbf{k})}{\partial k_y}\Big\rangle - \Big\langle \frac{\partial\psi(\mathbf{k})}{\partial k_y}\Big|\frac{\partial\psi(\mathbf{k})}{\partial k_x}\Big\rangle\Big]$$

For one band!

We define for the gap:

$$C_{\mathrm{gap}} = \sum_{n<\mathrm{gap}} C_n$$

Computing the Chern number confirms the topological behaviour. **Animation**

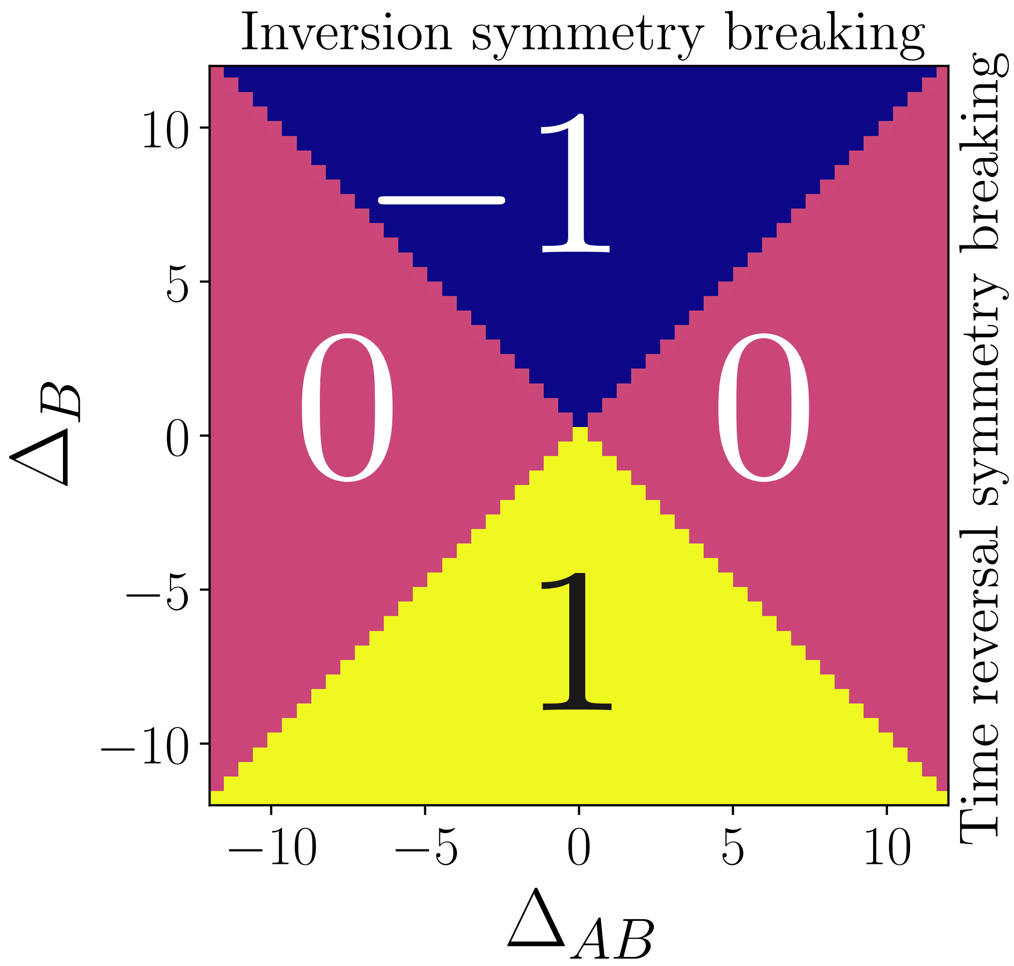
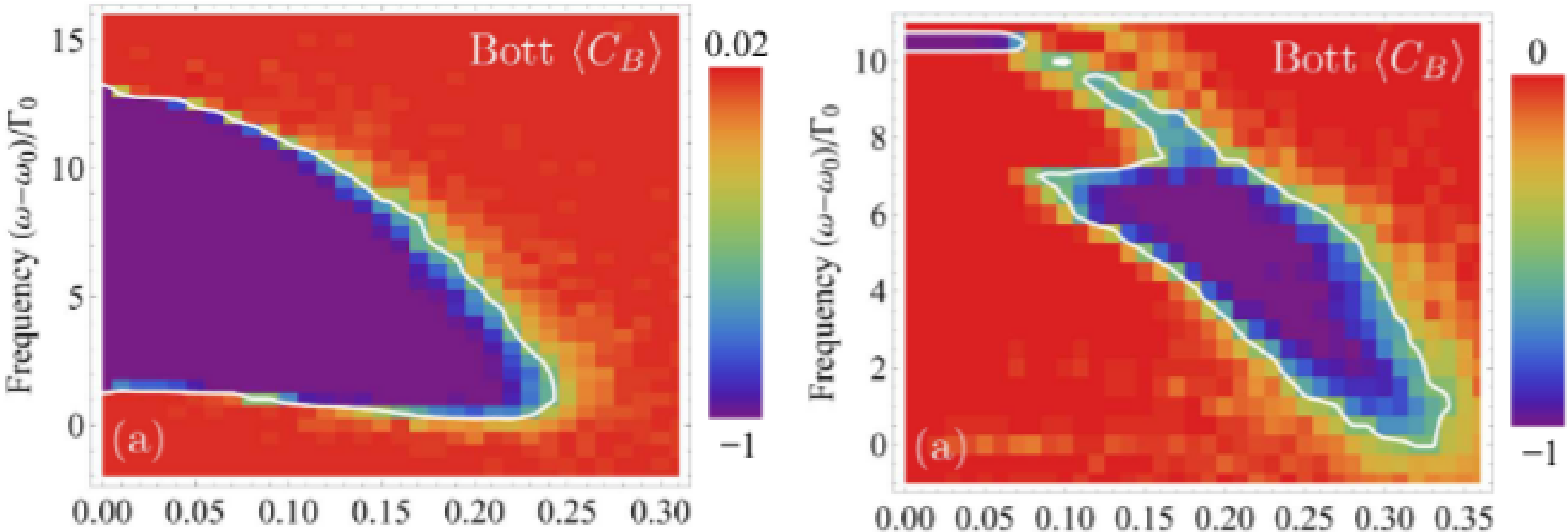

Inversion symmetry breaking
Time reversal symmetry breaking
−1
0
0
1
10
5
0
−5
−10
Δ_B
−10
−5
0
5
10
Δ_{AB}

Figure 11.



Topological transitions and Anderson localization of light in disordered atomic arrays

S.E. Skipetrov* and P. Wulles
Univ. Grenoble Alpes, CNRS, LPMMC, 38000 Grenoble, France
(Dated: April 8, 2022)

We explore the interplay of disorder and topological phenomena in honeycomb lattices of atoms coupled by the electromagnetic field. On the one hand, disorder can trigger transitions between distinct topological phases and drive the lattice into the topological Anderson insulator state. On the other hand, the nontrivial topology of the photonic band structure suppresses Anderson localization of modes that disorder introduces inside the band gap of the ideal lattice. Furthermore, we discover that disorder can both open a topological pseudogap in the spectrum of an otherwise topologically trivial system and introduce spatially localized modes inside it.

Figure 12.

Can we avoid using magnetic field ?

Deforming the cell

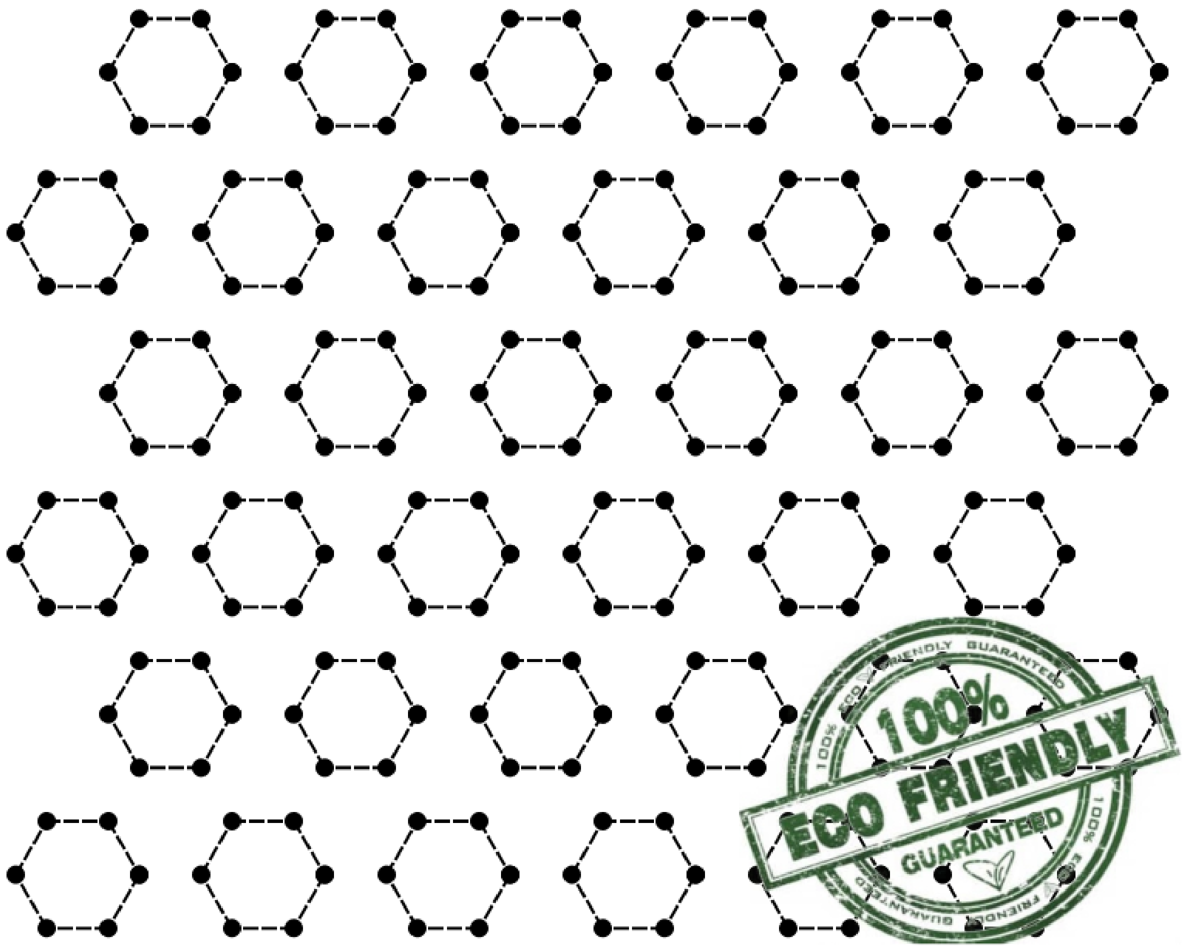


Figure 13. **Animation**

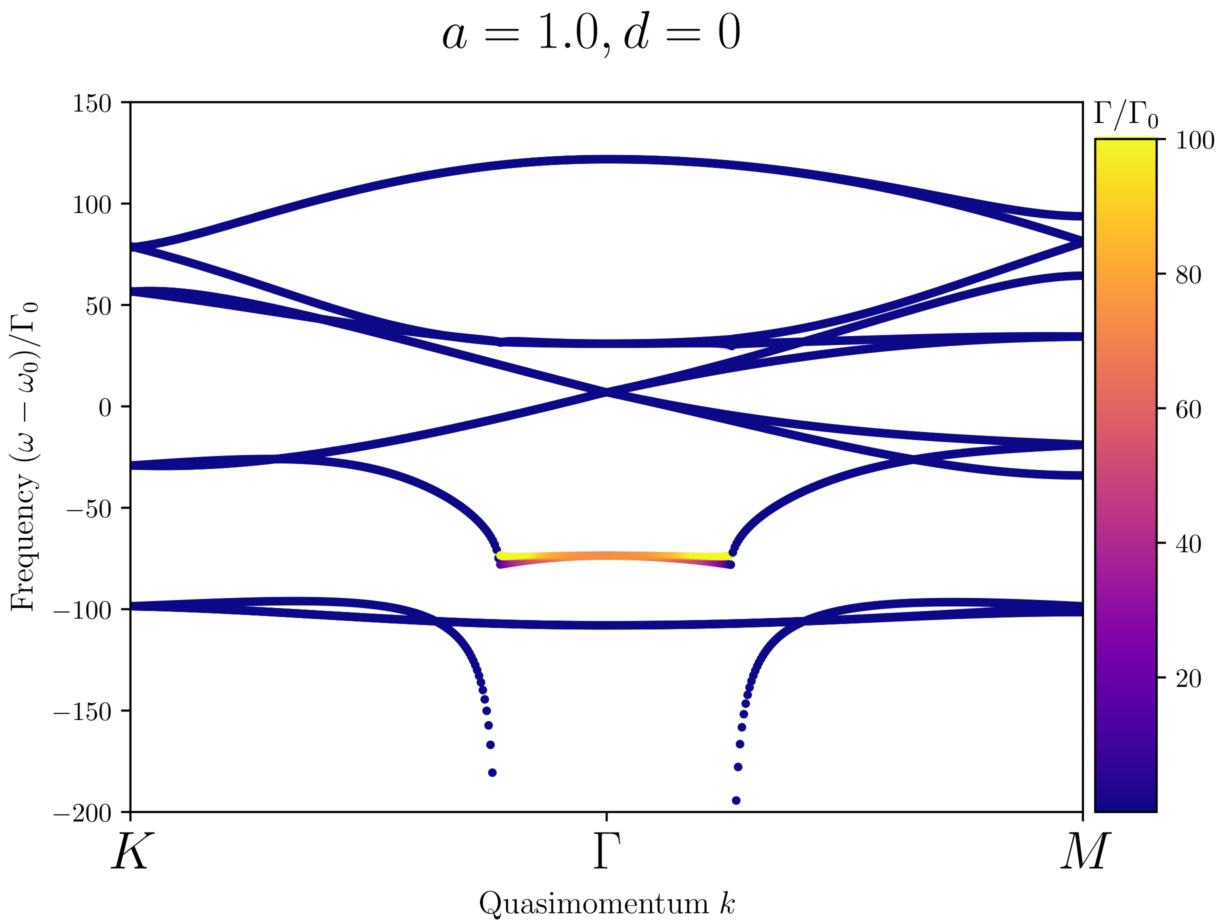

$a = 1.0, d = 0$
Frequency $(\omega - \omega_0)/\Gamma_0$
150
100
50
0
−50
−100
−150
−200
K
Γ
M
Quasimomentum k
Γ/Γ_0
100
80
60
40
20

Figure 14.

Folding in the Brillouin space

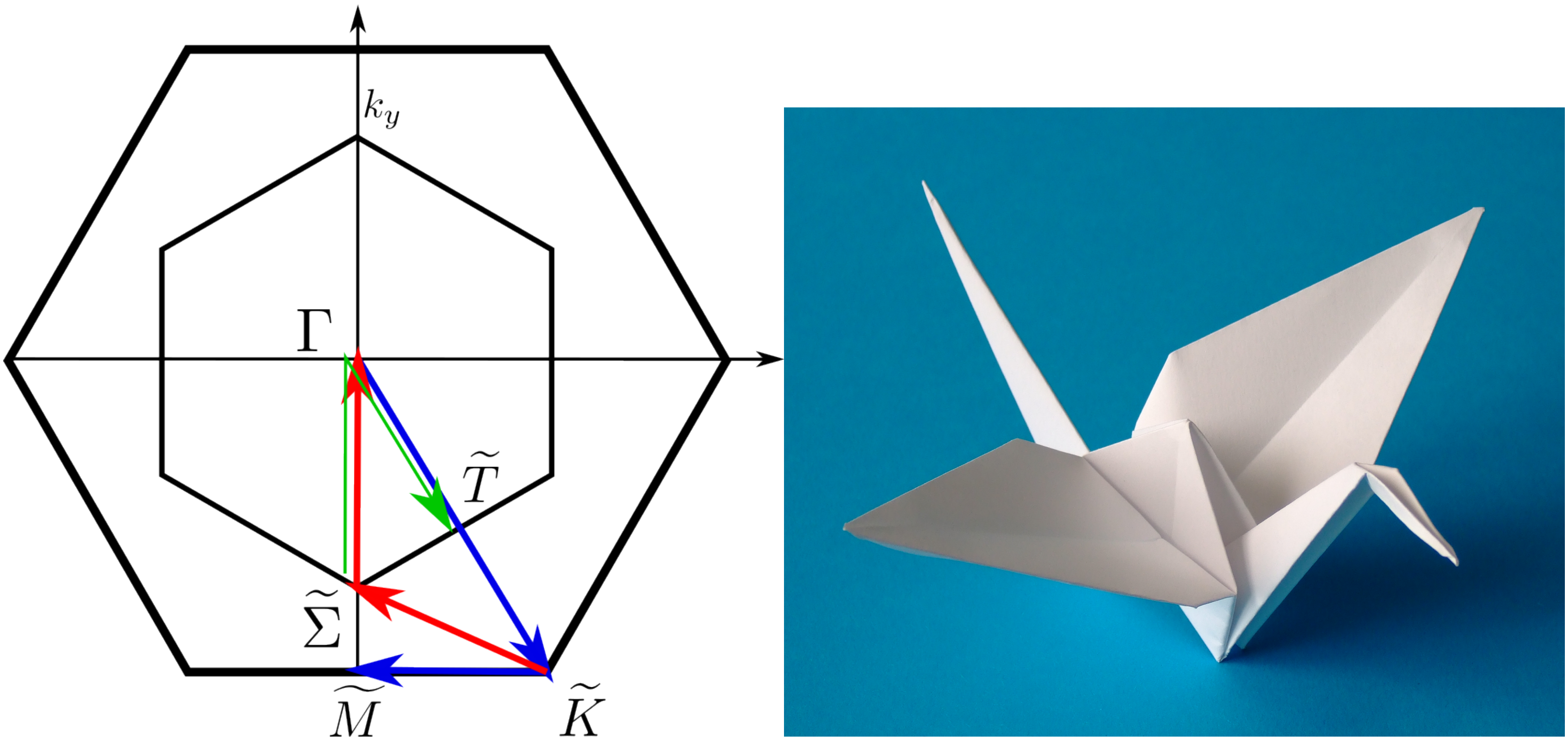


Figure 15. **Animation**

Gaps in the six-cell

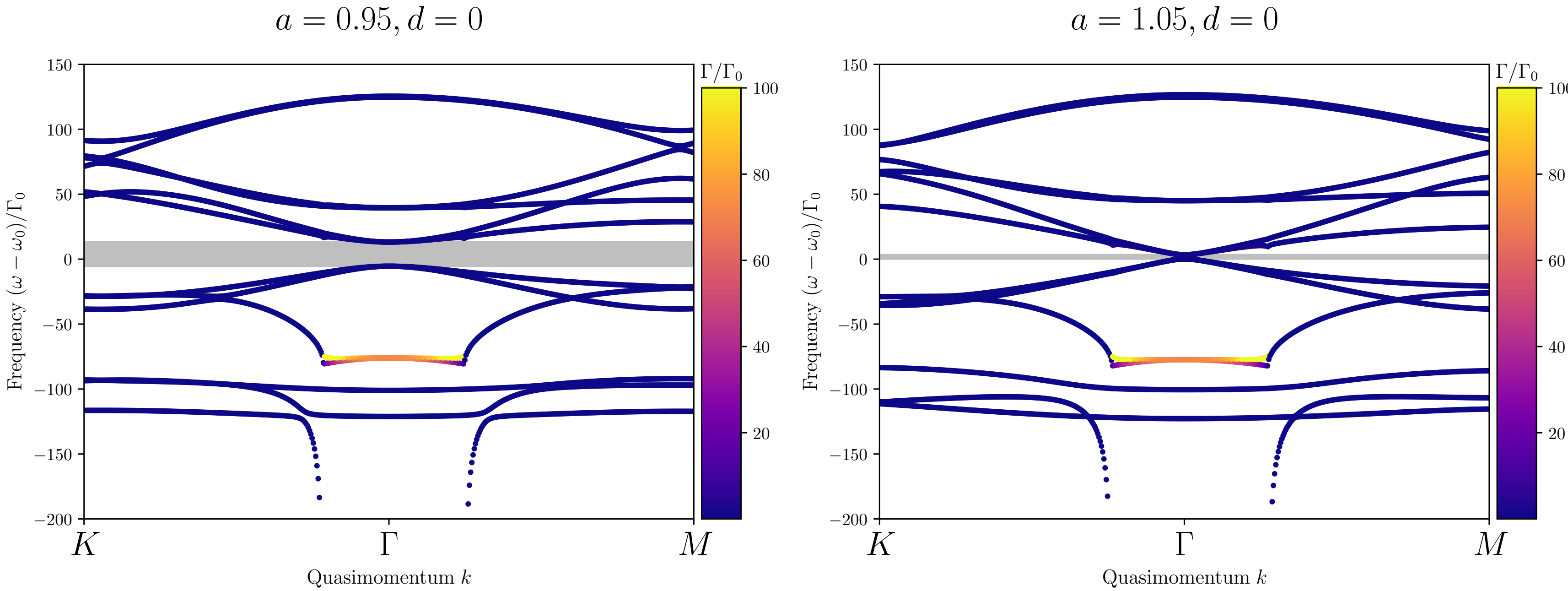


Figure 16.

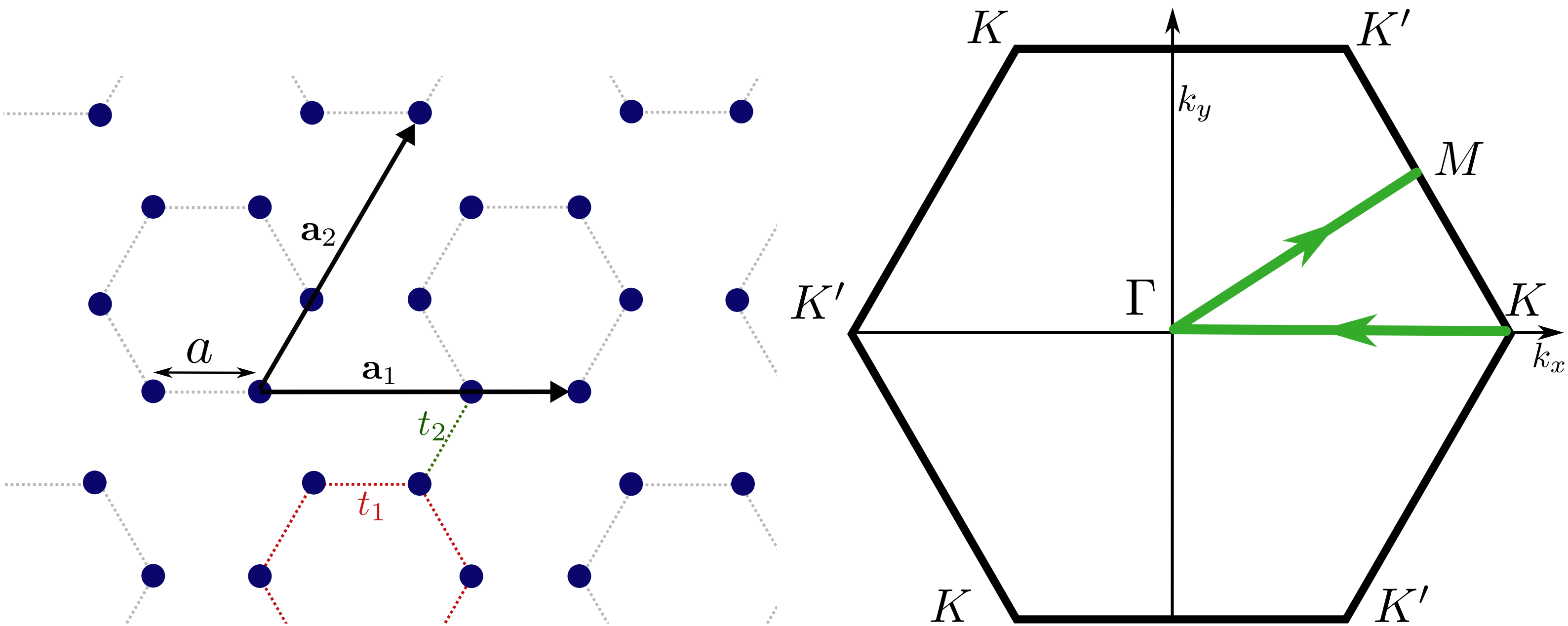


Figure 17.

- **Scheme for Achieving a Topological Photonic Crystal by Using Dielectric Material** – *Long-Hua Wu and Xiao Hu*
- **Two-Dimensionally Confined Topological Edge States in Photonic Crystals** – *Barik et al*

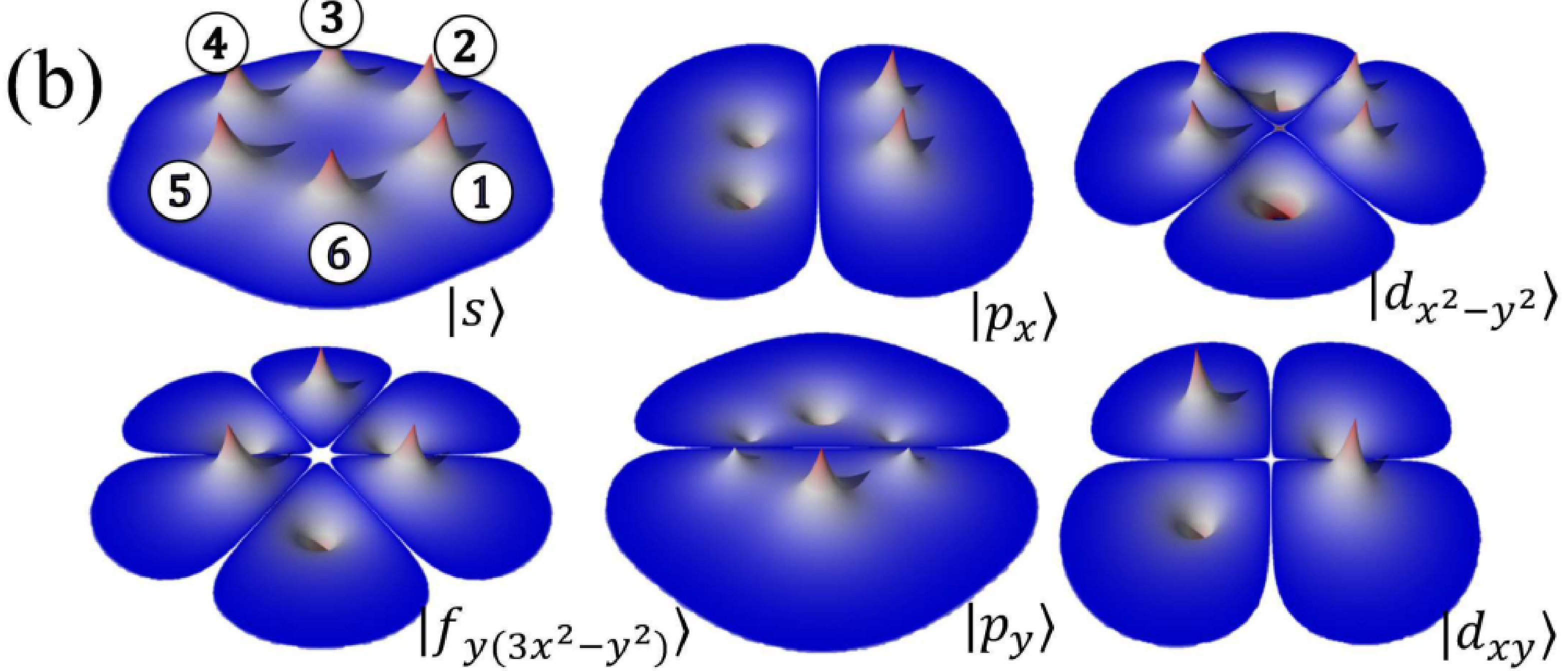


Figure 18. Taken from *Wu et al*

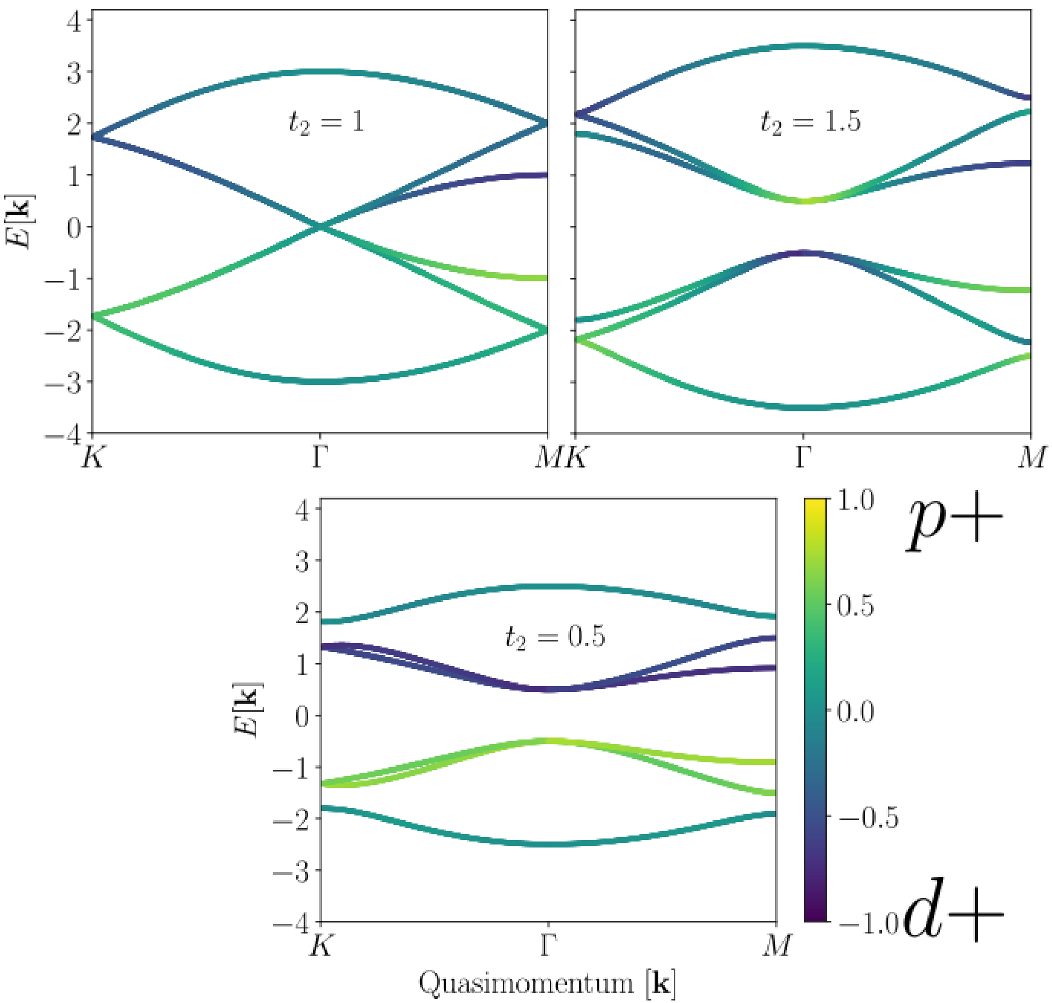

$t_2 = 1$
$t_2 = 1.5$
$t_2 = 0.5$
$E[\mathbf{k}]$
K
Γ
M
Quasimomentum $[\mathbf{k}]$
$p+$
$d+$
1.0
0.5
0.0
−0.5
−1.0

Figure 19.

Spin Chern number

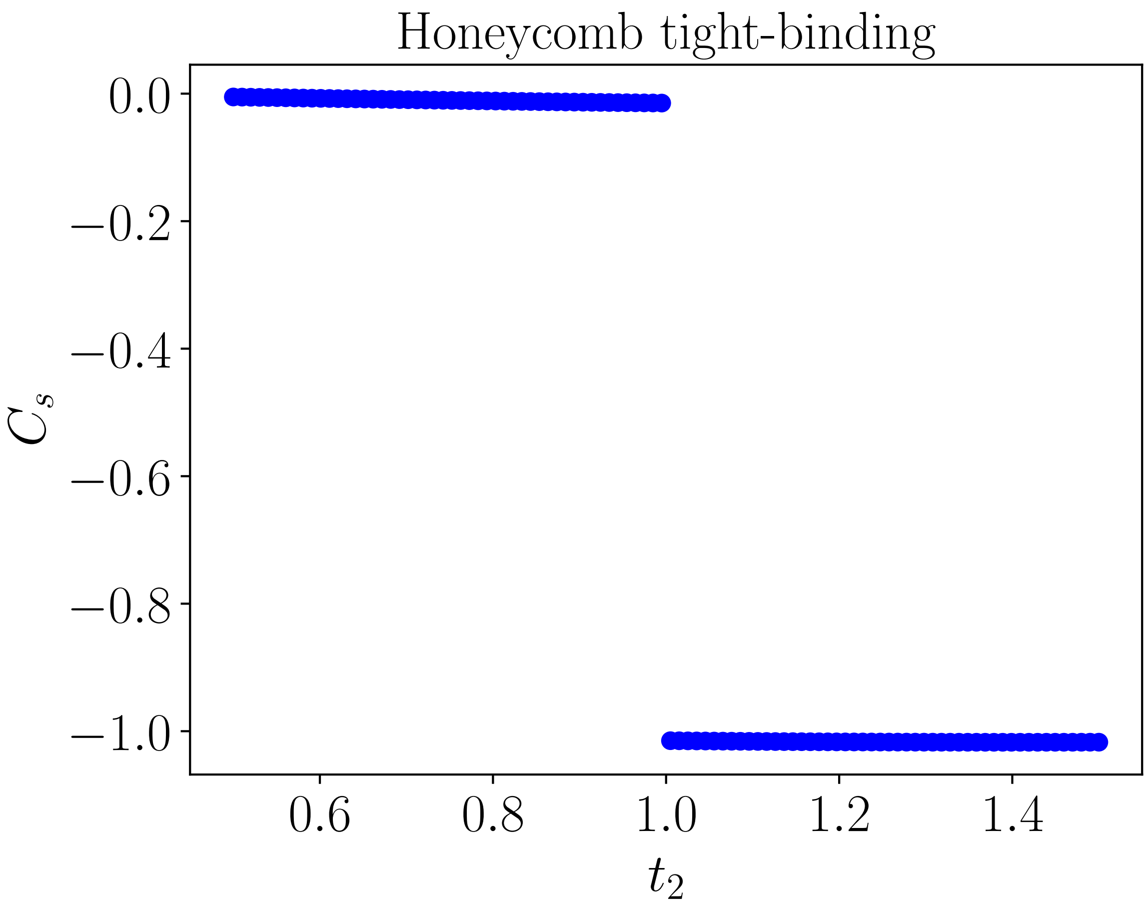


Figure 20.

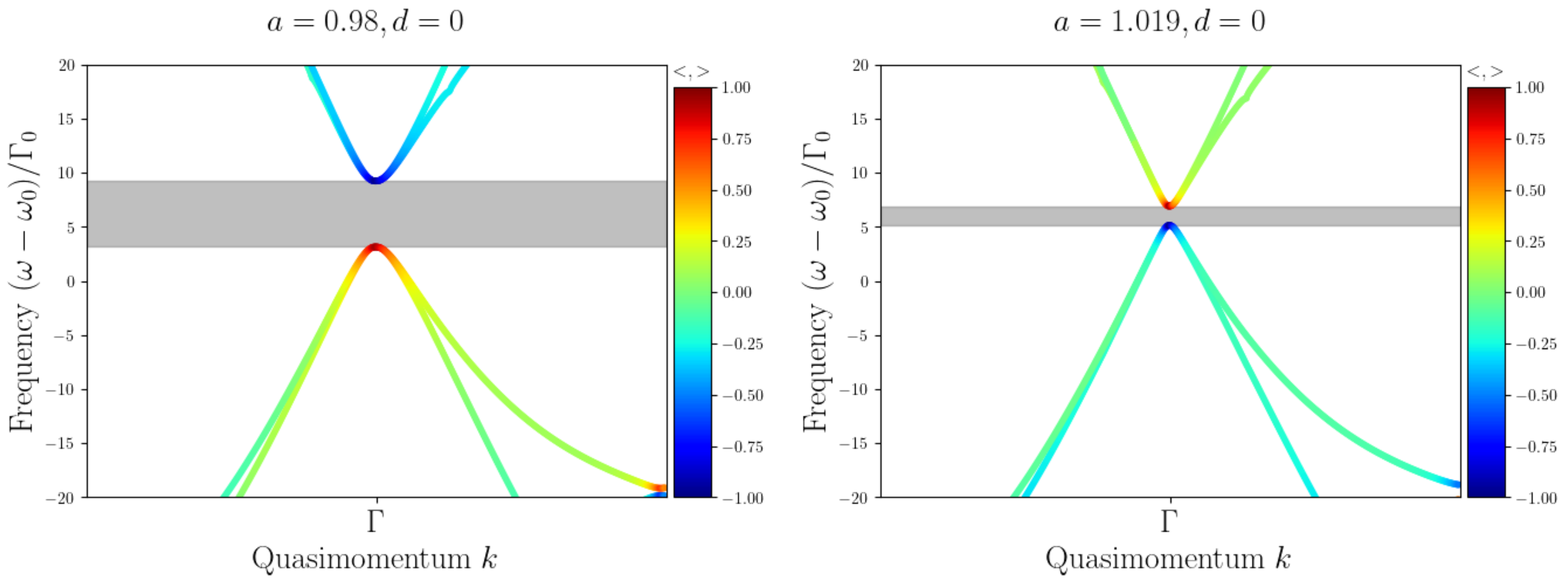


Figure 21.

Bien expliquer que ici les exciations ont des orientations en espace, comme au début

Conclusion

- Similar phenomena to the QHE or the QSHE have been recreated in photonic systems.
- The existence of a TAI has been demonstrated in the phononic context.
- A complete description of the gap in a honeycomb lattice system with a magnetic field and two types of sites, A and B, can be provided.
- We provided a theoretical background for the experiments conducted in Nice.

Questions ?

Invariance of scale

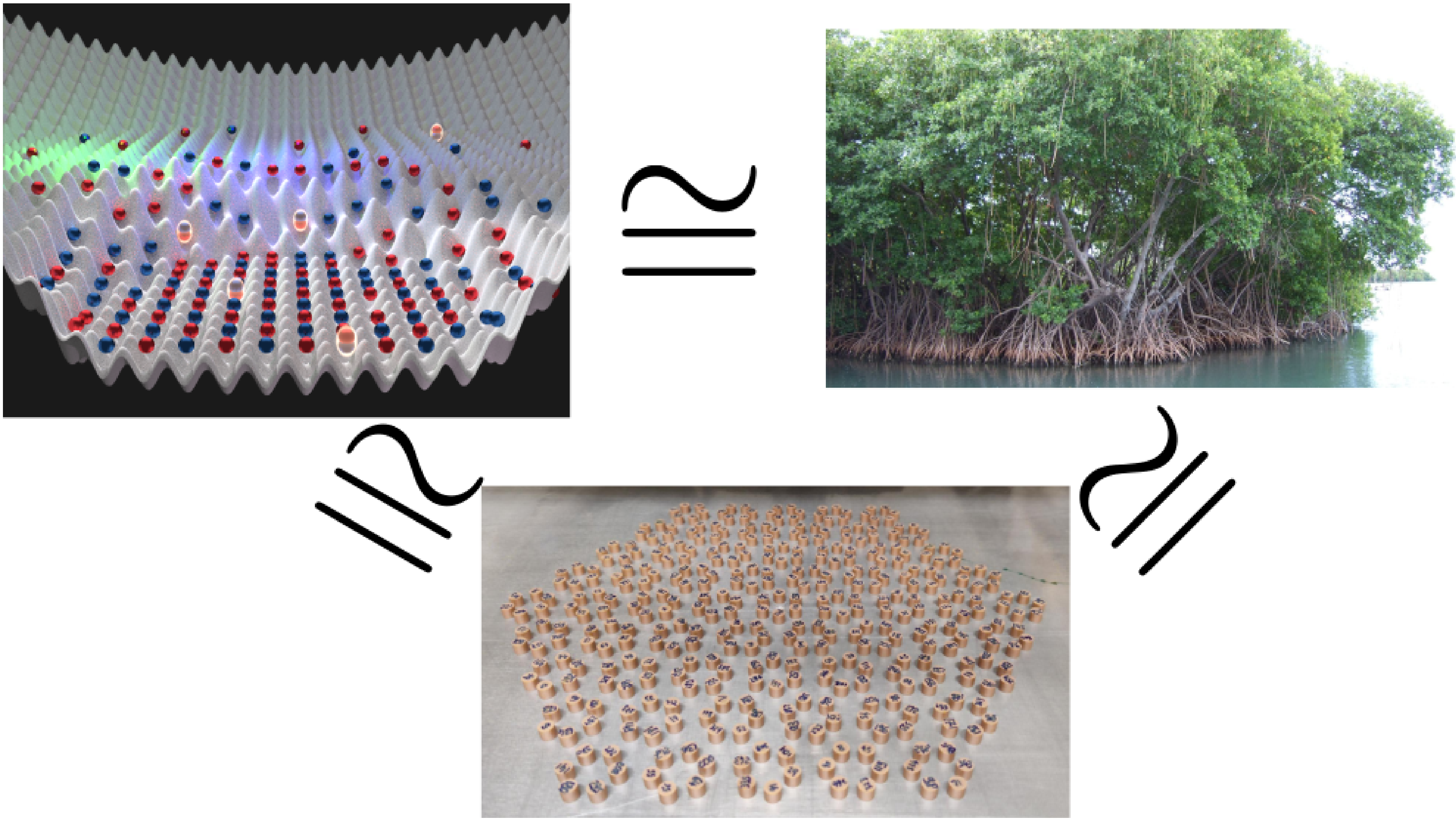


Figure 22.

Hamiltonian of the system

The hamiltonian of the atomic subsystem can be written as:

$$\begin{aligned} H &= \hbar \sum_{n=1}^{N} \sum_{\alpha_n = \sigma_{\pm,0}} \left(\omega_0 + \delta\omega_0^{(A,B)} + \mathrm{sgn}(\alpha_n)\mu_B B - i\frac{\Gamma_0}{2} \right) |\alpha_n\rangle\langle\alpha_n| \\ &+ \frac{3\pi\hbar\Gamma_0}{k_0} \sum_{n \neq m}^{N} \sum_{\alpha_n, \beta_m = \sigma_{\pm}, 0} \mathcal{G}_{\alpha\beta}(\mathbf{r}_n - \mathbf{r}_m) |\alpha_n\rangle\langle\beta_m| \end{aligned}$$

With:

- N the number of identical two-levels atoms.
- $\omega_0 + \delta\omega_0^{(A,B)}$ resonance frequency of atoms of the sublattice A or B
- $\boldsymbol{B}$ the static magnetic field
- $\mathcal{G}_{\alpha\beta}$ the dyadic Green's function describing the coupling of atoms by EM waves
- σ reprents the three polarisations $\pm$ in the plane and 0 perpendicular

We define a $3N \times 3N$ dimensionless matrix G made of 3×3 blocks:

$$G_{mn} = \delta_{mn}(i \pm 2\Delta_{AB} - 2\alpha\Delta_B)\mathbb{1}_{3\times 3} + (1-\delta_{mn})d_{\mathrm{eg}}A_{mn}d_{\mathrm{eg}}^{\dagger}$$

With:

- $\Delta_{AB} = (\delta\omega_0^B - \delta\omega_0^A)/2\Gamma_0$
- $\Delta_B = \mu B/\Gamma_0$ and α is -1,1, and 0 for the three diagonal elements.
- d_{eg} is a transformation matrix to go from cartesian to polar coordinates

And:

$$A_{mn} = (1-\delta_{mn})\frac{3}{2}\frac{\mathrm{e}^{ik_0 r}}{k_0 r}\left(P(i\,k_0 r)\mathbb{1} + Q(i\,k_0 r)\frac{\mathbf{r}_{mn}\otimes\mathbf{r}_{mn}}{r_{mn}^2}\right)$$

Where:

- $k_0 = \omega_0/c$
- $\mathbf{r}_{mn} = \mathbf{r}_m - \mathbf{r}_n$
- $P(x) = 1 - 1/x + 1/x^2$ and $Q(x) = -1 + 3/x - 3/x^2$

An eigenproblem

The problem is now to solve:

$$G\mathbf{\Psi} = \Lambda\mathbf{\Psi}$$

where $\mathbf{\Psi}$ are the quasimodes of the atomic system with eigenfrequencies $\omega = \omega_0 - \frac{\Gamma_0}{2}\Re(\Lambda)$ and decay rates $\Gamma = \Gamma_0\Im(\Lambda)$.

Using periodicity of the lattice and restricting ourselves to TE modes (Bloch theorem):

$$\mathbf{\Psi}_n = \psi(\mathbf{k})\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}_n}$$

Transforms the problem into:

$$M\psi(\mathbf{k}) = \Lambda\psi(\mathbf{k})$$

Where M is a 4×4 matrix:

$$M = \begin{pmatrix} S_1^{\sigma+\sigma+} & S_1^{\sigma+\sigma-} & S_2^{\sigma+\sigma+} & S_2^{\sigma+\sigma-} \\ S_1^{\sigma-\sigma+} & S_1^{\sigma-\sigma-} & S_2^{\sigma-\sigma+} & S_2^{\sigma-\sigma-} \\ S_3^{\sigma+\sigma+} & S_3^{\sigma+\sigma-} & S_4^{\sigma+\sigma+} & S_4^{\sigma+\sigma-} \\ S_3^{\sigma-\sigma+} & S_3^{\sigma-\sigma-} & S_4^{\sigma-\sigma+} & S_4^{\sigma-\sigma-} \end{pmatrix}$$

and:

$$\begin{aligned}
S_1^{\alpha\beta} &= \sum_{\mathbf{r}_m \in A} G_{AA}^{\alpha\beta}(\mathbf{r}_m) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}_m} \\
S_2^{\alpha\beta} &= \sum_{\mathbf{r}_m \in A} G^{\alpha\beta}(\mathbf{r}_m + \mathbf{a}_1) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}_m} \\
S_3^{\alpha\beta} &= \sum_{\mathbf{r}_m \in A} G^{\alpha\beta}(\mathbf{r}_m - \mathbf{a}_1) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}_m} \\
S_4^{\alpha\beta} &= \sum_{\mathbf{r}_m \in A} G_{BB}^{\alpha\beta}(\mathbf{r}_m) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}_m}
\end{aligned}$$

We stress that here:

$$\begin{aligned}
G_{AA}(0) &= \begin{pmatrix} i + 2\Delta_{AB} + 2\Delta_B & 0 \\ 0 & i + 2\Delta_{AB} - 2\Delta_B \end{pmatrix} \\
G_{BB}(0) &= \begin{pmatrix} i - 2\Delta_{AB} + 2\Delta_B & 0 \\ 0 & i - 2\Delta_{AB} - 2\Delta_B \end{pmatrix}
\end{aligned}$$

We can simplify these sums using Poisson summation formula:

$$\sum_{\substack{\mathbf{r}_m \in A \\ \mathbf{r}_m \neq 0}} G^{\alpha\beta}(\mathbf{r}_m)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}_m} = \frac{1}{\mathcal{A}} \sum_{\mathbf{g}_m \in A'} g^{\alpha\beta}(\mathbf{g}_m - \mathbf{k}; 0) - G^{\alpha\beta}(\mathbf{0})$$

where $\mathcal{A} = \frac{3\sqrt{3}}{2}\frac{1}{a^2}$ is the area of the Brillouin zone and $g = \mathfrak{F}(G)$ and thus:

$$g_{\alpha\beta}(\mathbf{q}; \mathbf{r}') = \frac{(k_0^2\,\delta_{\alpha\beta} - q_\alpha q_\beta)}{2\pi k_0^2}\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}'}\int \frac{1}{k_0^2 - \mathbf{q}^2 - q_z^2}\mathrm{d}q_z$$

To ensure convergence of this integral we introduce a gaussian cut-off :

$$g^*_{\alpha\beta}(\mathbf{q}; \mathbf{r}') = \frac{(k_0^2\,\delta_{\alpha\beta} - q_\alpha q_\beta)}{2\pi k_0^2}\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}'}\int \frac{\mathrm{e}^{-a_{\mathrm{ho}}^2(\mathbf{q}^2 + q_z^2)/2}}{k_0^2 - \mathbf{q}^2 - q_z^2}\mathrm{d}q_z$$

Topological Quantum Optics in Two-Dimensional Atomic Arrays, *PRL 119, J. Perczel, and M. D. Lukin*

Which can be expressed in a closed form:

$$\boxed{g^*_{\alpha\beta}(\mathbf{q};0) = (\delta_{\alpha\beta}k_0^2 - q_\alpha q_\beta)\mathcal{I}(\mathbf{q})}$$

With:

$$\begin{aligned}
\mathcal{I}(\mathbf{q}) &= \chi(\mathbf{q})\frac{\pi}{\Lambda(\mathbf{q})}(-i + \mathrm{erfi}(a_{\mathrm{ho}}\Lambda(\mathbf{q})/\sqrt{2})) \\
\chi(\mathbf{q}) &= \frac{1}{2\pi k_0^2}\mathrm{e}^{-(k_0 a_{\mathrm{ho}})^2/2} \\
\Lambda(\mathbf{q}) &= \sqrt{k_0^2 - q^2}
\end{aligned}$$

And similarly we can compute $G^*_{\alpha\beta}(\mathbf{0})$ as the inverse Fourier transform of $g^*_{\alpha\beta}(\mathbf{q};0)$ where $\mathbf{q}=0$

$$\boxed{G^*_{\alpha\beta}(\mathbf{0}) = \frac{k_0}{6\pi}\left(\frac{\mathrm{erfi}(k_0 a_{\mathrm{ho}}/\sqrt{2}) - i}{\mathrm{e}^{k_0^2 a_{\mathrm{ho}}^2/2}} - \frac{-1/2 + (k_0 a_{\mathrm{ho}})^2}{\sqrt{\pi/2}(k_0 a_{\mathrm{ho}})^3}\right)\delta_{\alpha\beta}}$$

Adding plates

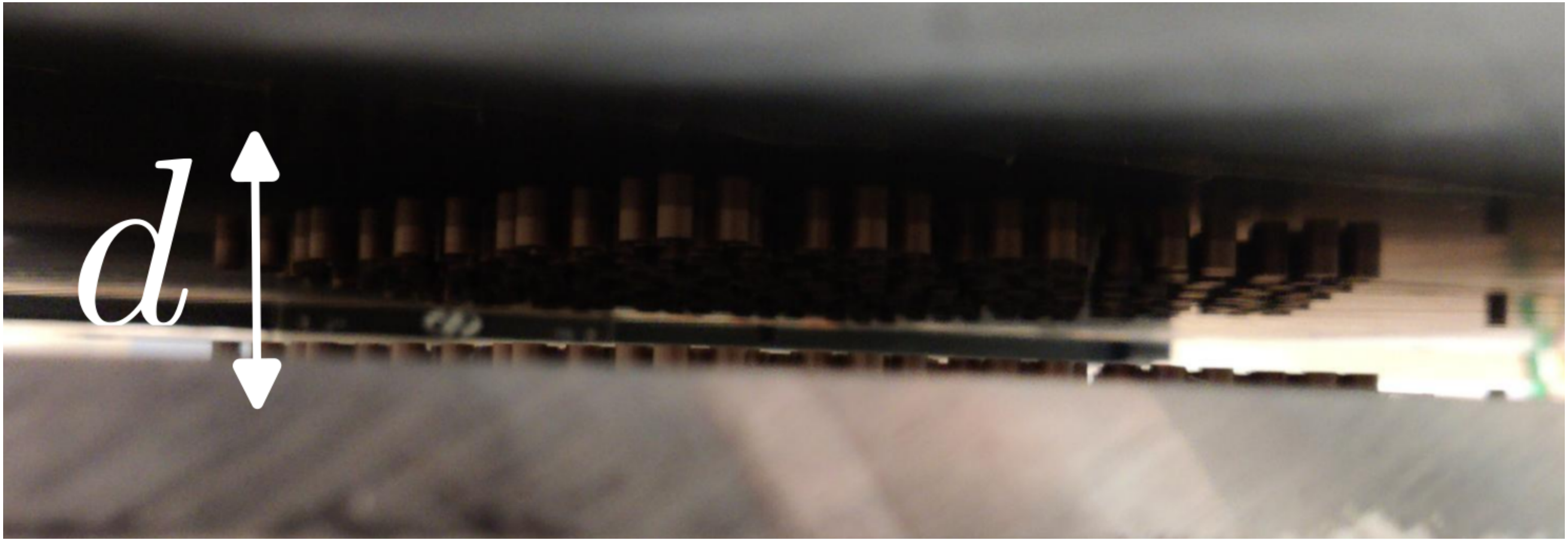


Figure 23.

- Conforms to experiment
- Makes our model hermitian

Adding plates

Method of images:

$$g^*_{\alpha\beta}(\mathbf{q};0) = (\delta_{\alpha\beta}k^2 - q_\alpha q_\beta)\left(\mathcal{I}(\mathbf{q}) + \frac{i}{\sqrt{k_0^2 - q^2}k_0^2}\frac{1}{1 + \mathrm{e}^{-i\sqrt{k_0^2-q^2}d}}\right), \sqrt{k_0^2 - q^2}d \neq \pi$$

$$\begin{aligned} G^*_{\text{plates}}(\mathbf{d}) &= \frac{k_0}{6\pi}\left(\frac{\mathrm{erfi}(k_0\,a/\sqrt{2}) - i}{\mathrm{e}^{k_0^2a^2/2}} - \frac{-1/2 + (k_0\,a)^2}{\sqrt{\pi/2}(k_0\,a)^3}\right. \\ &- \left. 3\left(\frac{\ln(1+\mathrm{e}^{ik_0d})}{k_0d} + i\frac{\mathrm{Li}_2(-\mathrm{e}^{ik_0d})}{(k_0d)^2} - \frac{\mathrm{Li}_3(-\mathrm{e}^{ik_0d})}{(k_0d)^3}\right)\delta_{\alpha\beta}\right) \end{aligned}$$

Band diagram for $d \neq 0$

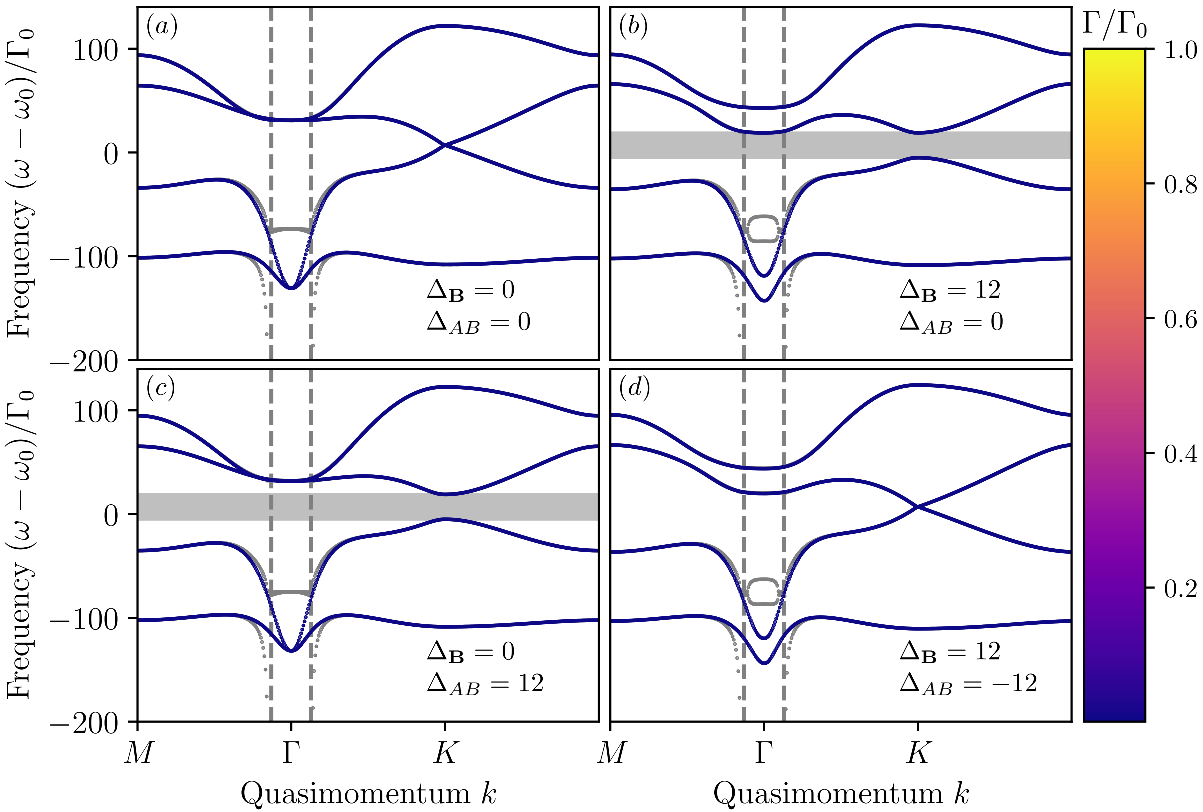


Figure 24.

A compact formula

$$
\begin{aligned}
W_{\text{gap}} &= 2||\Delta_{\mathbf{B}}| - |\Delta_{AB}|| && \text{if} \quad 0 < |\Delta_{\mathbf{B}}| < R_1 \\
&= \frac{1}{2}(c_0 - c_1 + S - |\Delta_{AB}|) && \text{if} \quad R_1 < |\Delta_{\mathbf{B}}| < R_2 \\
&= -2|\Delta_{\mathbf{B}}| + S && \text{if} \quad R_2 < |\Delta_{\mathbf{B}}| < R_3 \\
&= 0 && \text{elsewhere}
\end{aligned}
$$

Where:

$$
S = 2\mathrm{Re}\left(\sqrt{4\Delta_{AB}^2 + \frac{1}{4}(c_2 + i\, c_3)^2}\right)
$$

$$
R_1 = \frac{1}{4}(c_0 - c_1 + S + 2|\Delta_{AB}|)
$$

$$
R_2 = \frac{1}{4}(c_0 - c_1 + S - 2|\Delta_{AB}|)
$$

$$
R_3 = S/2
$$

c_0,c_1 and c_2 are elements of the matrix $M(\mathbf{k})$ and they only rely on $k_0 a$, the width of the gap is dependant of three parameters: $\Delta_{\mathbf{B}}$,Δ_{AB} and $k_0 a$.

Spin Chern number (evil technical details)

$$H \rightarrow H' = P\sigma P$$

Where P projects on eigenstates below the gap and:

$$\sigma = \mathrm{diag}(0, 1, -1, 1, -1, 0)$$

Then:

$$C_{\pm} = \frac{1}{2\pi i} \iint_{\mathrm{BZ}} F_{12}^{\pm}(k)$$

And:

$$C_s = \frac{1}{2}(C_{+} - C_{-})$$